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Critical dynamics of the open Ising chain

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The one-dimensional ferromagnetic Ising model with open boundary conditions within the Glauber dynamics is studied. From the behavior of the order parameter, the explicit scaling form of the relaxation time is obtained. For finite systems, the dynamical critical exponent is $z = 1$, in contrast to the value of the infinite open chain ($z = 2$).

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The universality of the dynamical critical exponent z for the one-dimensional Ising model within Glauber dynamics has been considered in the past [1–4]. For the ferromagnetic chain it is well known that $z = 2$. This exponent can vary by controlling a parameter in the transition rate entering the master equation [2]. Moreover, for the alternating ferromagnetic chain z depends strongly on the coupling constants [3,4]. All these results have been obtained for the *closed* chain. In this paper we consider the critical dynamics of the ferromagnetic *open* Ising chain. We obtain explicitly the scaling form of the relaxation time of the magnetization of finite chains. We find that $z = 2$ only when the limit of infinite chain is taken. For finite chains we show that the relaxation time diverges with $z = 1$.

The model is described by the Hamiltonian for N spins,

$$-\frac{H}{k_BT} = \sum_{i=1}^{N-1} K s_i s_{i+1} \ , \tag{1}$$

where $K = J/k_BT$ with $J > 0$, $s_i = \pm 1$, and the sum runs over the lattice sites i. The static solution is easily obtained for the open chain and the relevant results for this paper are the following: (a) the transition temperature is $T_c = 0$ and (b) the correlation length ξ near T_c behaves as $\xi \sim \exp(2K)/2$.

The master equation for the time evolution of the probability $P(s_1, s_2, \ldots, s_N; t)$ to find the system in the configuration $\{s\}$ at time t is

$$\begin{aligned}\frac{d}{dt}P(s_1,\ldots,s_N;t) &= -\sum_{i=1}^{N} w_i(s_i)P(s_1,\ldots,s_N;t)\\ &\quad+\sum_{i=1}^{N} w_i(-s_i)P(s_1,\ldots,-s_i,\ldots,s_N;t) \ . \end{aligned}\tag{2}$$

Here $w_i(s_1, s_2, \ldots, s_i, \ldots, s_N) = w_i(s_i)$ is the transition probability per unit time that the ith spin flips from the value s_i to $-s_i$ while all others are unaffected. These transition rates must satisfy the detailed balance condition [1,2]. In the present situation, the transition rates for the end spins s_1 and s_N and for the bulk spins of the lattice s_i ($1 < i < N$) are different. For the end spins the Glauber rates are

$$w_1(s_1) = \frac{1}{2}(1 - \beta s_1 s_2) \ , \tag{3}$$

$$w_N(s_N) = \frac{1}{2}(1 - \beta s_{N-1} s_N) \ , \tag{4}$$

with $\beta = \tanh(K)$. For the bulk spins the Glauber rates are

$$w_i(s_i) = \frac{1}{2}[1 - \gamma s_i(s_{i-1} + s_{i+1})] \ , \tag{5}$$

where $\gamma = (1/2)\tanh(2K)$. Note that the time scale of the thermal bath was incorporated in the time variable.

The evolution of the local magnetization $\langle s_i(t)\rangle = \sum_{\{s\}} s_i P(\{s\}, t)$ is described by

$$\frac{d}{dt}\langle s_i(t)\rangle = -2\langle s_i w_i(s_i)\rangle \ . \tag{6}$$

From Eq. (6) and using Eqs. (3)–(5), we obtain

$$\frac{d}{dt}\vec{S} = -\mathbf{M}\,\vec{S} \ , \tag{7}$$

where $\vec{S}$ is the N-dimensional column vector with the elements $\langle s_1(t)\rangle, \ldots, \langle s_N(t)\rangle$ and $\mathbf{M}$ is the square matrix

$$\mathbf{M} = \begin{pmatrix} 1 & -\beta & 0 & 0 & \ldots & 0 & 0 \\ -\gamma & 1 & -\gamma & 0 & \ldots & 0 & 0 \\ 0 & -\gamma & 1 & -\gamma & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & & & \\ 0 & 0 & 0 & 0 & \ldots & -\beta & 1 \end{pmatrix} . \tag{8}$$

In order to solve this dynamical equation, we must obtain the eigenvalues and eigenvectors of the tridiagonal matrix $\mathbf{M}$. Notice that the magnetization is not an eigenmode of the dynamics. However, it can be expressed as a linear combination of the eigenvectors of the matrix $\mathbf{M}$. Every eigenvector ϕ_j decays exponentially as $\exp(-\lambda_j t)$, where λ_j is the corresponding eigenvalue. The longest time behavior of the dynamics is dominated by the slow-

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est mode, say ϕ_1, if its amplitude is finite. Moreover, as the critical temperature is approached the magnetization coincides with the slowest mode ϕ_1. This is due to the fact that the amplitudes of all other modes that contribute to the magnetization vanish as $T \to T_c$. Therefore, we will focus our attention on the determination of the eigenvalue λ_1. The eigenvalues of the $N \times N$ matrix $\mathbf{M}$ are obtained from det $[\mathbf{M} - \lambda \mathbf{I}_N] = 0$, where $\mathbf{I}_N$ is the $N \times N$ identity matrix. The determinant can be evaluated by the cofactor expansion of the first two lines and the last two lines. We obtain

$$(1-\lambda)^2 \det [\mathbf{M}_{N-2}] - 2\gamma\beta \det [\mathbf{M}_{N-3}] + \gamma^2\beta^2 \det [\mathbf{M}_{N-4}] = 0 \quad , \quad (9)$$

where $\mathbf{M}_n$ is an $n \times n$ tridiagonal matrix with elements $M_{i,i} = 1 - \lambda$, $M_{i+1,i} = M_{i,i+1} = -\gamma$. Those matrices can be written as $\mathbf{M}_n = (1-\lambda)\mathbf{I}_n - \gamma \mathbf{V}_n$, where $\mathbf{V}_n$ is an $n \times n$ tridiagonal matrix with the diagonal elements (i,i) equal to zero and the elements $(i, i+1)$ and $(i+1, i)$ equal to 1. The eigenvalues of $\mathbf{V}_n$ are $v_r^n = 4\sin^2[\pi r/(2n+2)] - 2$ for $r = 1, 2, \ldots, n$. So the eigenvalues of $\mathbf{M}$ are given by

$$(1-\lambda)^2 F(\lambda, N-2) - 2\gamma\beta(1-\lambda)F(\lambda, N-3) + \gamma^2\beta^2 F(\lambda, N-4) = 0 \quad , \quad (10)$$

where

$$F(\lambda, n) = \prod_{i=1}^{n} \left\{ (1-\lambda) - \gamma \left[4\sin^2\left(\frac{\pi i}{2n+2}\right) - 2 \right] \right\} \quad . \quad (11)$$

We have studied the polynomial, for finite N, in the limits $\lambda \to 0$ and $T \to 0$. The general behavior of the smallest eigenvalue for an open chain of N sites is

$$\lambda_1 \sim \frac{2}{N-1}\exp(-2K) + \frac{4N(N-2)-6}{3(N-1)^2}\exp(-4K) + O\Big(\exp(-6K)\Big) \quad . \quad (12)$$

Therefore, for large N, we have, for the relaxation time τ_N,

$$\tau_N = \frac{1}{\lambda_1} = \frac{N\xi}{1 + \frac{N}{3\xi}} \quad , \quad (13)$$

where ξ is the correlation length of the infinite system. The thermodynamic limit of the relaxation time is $\tau_\infty = 3\xi^2$. Since $\tau_\infty \sim \xi^z$ [5], we have $z = 2$. Finite size scaling [6] states that, near T_c and for large N, $\tau_N/\tau_\infty = f(N/\xi)$. From Eq. (13) we see this expected behavior, namely,

$$\frac{\tau_N}{\tau_\infty} = \frac{N\xi^{-1}}{3 + N\xi^{-1}} \quad . \quad (14)$$

At fixed temperature and $N \to \infty$ $(N/\xi \gg 1)$ we have, as usual, $\tau_N/\tau_\infty \to 1$. Let us consider the finite chain, namely, N fixed and $\xi \to \infty$ $(N/\xi \ll 1)$. In general, we expect that τ_N has a nonsingular behavior and that $\tau_N/\tau_\infty \sim N^z\xi^{-z}$. Note that this determines the scaling relation $\tau_N/\tau_{N'} = (N/N')^z$. However, we obtain from Eqs. (13) and (14) that $\tau_N \sim N\xi$, implying that τ_N diverges at zero temperature with $z = 1$ and that $\tau_N/\tau_{N'} = (N/N')^z$. This anomalous scaling is related to the fact that the critical temperature is zero. It is worth mentioning that the *closed* finite chain does not show a finite size scaling form for the relaxation time. For this case the relaxation time of the magnetization does not depend on N and is proportional to ξ^2, implying $z = 2$ for any value of N.

Our results for the open chain can be explained as follows. When the chain is finite, the end spins are flipped with a larger rate ($w \sim \xi^{-1}$) than the inner spins ($w \sim \xi^{-2}$). This is an extra channel for the relaxation. In the thermodynamic limit the end spins are irrelevant and the system presents the usual way of relaxation, namely, the random walk of the domain walls [7].

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[1] R. J. Glauber, J. Math. Phys. **4**, 294 (1963).
[2] F. Haake and K. Thol, Z. Phys. B **40**, 219 (1980).
[3] M. Droz, J. Kamphorst Leal da Silva, and A. Malaspinas, Phys. Lett. A **115**, 448 (1986).
[4] James H. Luscombe, Phys. Rev. B **36**, 501 (1987).
[5] P. C. Hohenberg and B. I. Halperin, Rev. Mod. Phys. **49**, 435 (1977).
[6] E. Brézin, in *Finite-size Scaling*, edited by J. L. Cardy (North-Holland, Amsterdam, 1988), p. 12.
[7] R. Cordery, S. Sarker, and J. Toboshnik, Phys. Rev. B **24**, 5402 (1981).